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Journal of Applied Logic 3 (2005) 43–65
JOURNAL OF APPLIED LOGIC
www.elsevier.com/locate/jal

On the structure of paraconsistent extensions of Johansson's logic ☆

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Available online 20 August 2004

Abstract

The aim of this article is to give a compact and self-contained description of the class of paraconsistent extensions of Johansson's (or minimal) logic (denoted **Lj**). The class of all non-trivial **Lj**-extensions is divided into three classes: the class **Int** of intermediate logics, the class **Neg** of negative logics (with axiom $\neg p$), and the class **Par** of proper paraconsistent **Lj**-extensions. For elements of **Par**, we define their intuitionistic and negative counterparts from classes **Int** and **Par**, respectively, and study to which extend paraconsistent logics are determined by their counterparts. To this end we need special presentation of j-algebras, which is also given in the article. In conclusion, we study Kripke semantics for paraconsistent **Lj**-extensions.

Keywords: Minimal logic; Paraconsistent logic; Algebraic semantics; j-algebra; Kripke semantics

1. Introduction

The aim of this article is to describe the structure of the class of paraconsistent extensions of Johansson's (or minimal) logic (denoted **Lj**). Some results of the article were included in my reports at the earlier paraconsistent meetings (see [6–8]), but the material was divided into parts and mixed with other topics. Here I will present all this material in a systematic form adding also the new results about interrelations between the intervals of

☆ The investigations presented in the article were supported by Alexander von Humboldt Foundation.
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1570-8683/$ – see front matter
doi:10.1016/j.jal.2004.07.011

the form $Spec(I, N)$ (Section 4), about the cardinality of intervals of this form (Section 6), and about the Kripke semantics for paraconsistent **Lj**-extensions (see Section 7).

Of course, the title of the article, which combines terms "paraconsistent" and "Johansson's logic", needs some explanations. The definition of paraconsistent logics as logics admitting inconsistent non-trivial theories is usually accompanied with the commentary that not all logics satisfying this definition are "really" paraconsistent, for example, Johansson's logic, because in **Lj** any negated formula is inferable from contradiction. However, there are several reasons that justify this combination of terms and this topic of investigations. First, Johansson's logic is worthy of interest as paraconsistent analogue of intuitionistic logic. Second, the study of lattices of logics, for example, intermediate or modal logics, plays a very important role in the development of non-classical logics. And third, the paraconsistent point of view, namely, paying attention to inconsistent theories and non-equivalent contradictions, takes a central place in our investigation.

2. Preliminary remarks

The negation in minimal logic **Lj** as well as in any of its extensions can be defined as reduction to a propositional constant $\bot$, "absurdity". Therefore, we choose a propositional language $\{\wedge, \vee, \to, \bot\}$ as basic and consider a negation as an abbreviation, $\neg\varphi := \varphi \to \bot$. We will use also the abbreviation $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$. As usual, by a *logic* we mean a set of formulas closed under substitution and *modus ponens*. For a logic L and a set of formulas X, $L + X$ denotes the least logic containing L and all formulas of X. With any logic L, we associate in a standard way an inference relation $\vdash_L$. For a set of formulas X and a formula φ, the relation $X \vdash_L \varphi$ means that φ can be obtained from elements of X and tautologies of L in a finite number of steps by using the rule of *modus ponens*.

A logic L is said to be *explosive* if the associated consequence relation possesses the property that $\{\varphi, \neg\varphi\} \vdash_L \psi$ for any φ and ψ. A *paraconsistent* logic is non-explosive.

The minimal or Johansson logic **Lj** can be defined in the chosen language via only positive axioms leaving the constant $\bot$ undefined:

(1) $p \to (q \to p)$,
(2) $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$,
(3) $(p \wedge q) \to p$,
(4) $(p \wedge q) \to q$,
(5) $(p \to q) \to ((p \to r) \to (p \to (q \wedge r)))$,
(6) $p \to (p \vee q)$,
(7) $q \to (p \vee q)$,
(8) $(p \to r) \to ((q \to r) \to ((p \vee q) \to r))$.

The same axioms will define in the positive language $\{\wedge, \vee, \to\}$ the well known positive logic **Lp**. We denote $\mathbf{Jhn}^+$($\mathbf{Jhn}$) the class of all (non-trivial) extensions of Johansson's logic. The join operation in the lattice of logics $\mathbf{Jhn}^+$ we denote $\vee$, the meet operation, which coincides with the set-theoretical intersection, we denote as usual $\cap$.

Below we introduce the denotation for several important **Lj**-extensions. In the choice of denotation, we follow [11].

- $\mathbf{Li} = \mathbf{Lj} + \{\bot \to p\}$ is *intuitionistic logic*.
- $\mathbf{Ln} = \mathbf{Lj} + \{\bot\}$ is *minimal negative logic*.
- $\mathbf{Le} = \mathbf{Lj} + \{p \vee (p \to q)\}$ is *Curry's logic of classical refutability*.
- $\mathbf{Lk} = \mathbf{Li} + \{p \vee (p \to q)\}$ is *classical logic*.
- $\mathbf{Lmn} = \mathbf{Ln} + \{p \vee (p \to q)\}$ is *maximal negative logic*.
- F is *trivial logic*, i.e., the set of all formulas.

Despite the fact that we postulate nothing about the constant $\bot$, the negation defined trough $\bot$ shares many important properties of intuitionistic and classical negations.

Proposition 1. *The following formulas are provable in* **Lj**:

(1) $\neg\neg(p \vee \neg p)$,
(2) $(p \to \neg q) \to (q \to \neg p)$,
(3) $(p \to q) \to (\neg q \to \neg p)$,
(4) $(\neg p \vee \neg q) \to \neg(p \wedge q)$,
(5) $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$,
(6) $p \to \neg\neg\neg p$,
(7) $\neg\neg\neg p \leftrightarrow \neg p$,
(8) $\neg(p \wedge \neg p)$,
(9) $(p \vee q) \to \neg(\neg p \wedge \neg q)$,
(10) $(p \wedge q) \to \neg(\neg p \vee \neg q)$,
(11) $(p \to q) \to \neg(p \wedge \neg q)$.

Below we give a few definitions and facts concerning the algebraic semantics of propositional logics. The detailed information can be found in [10,11].

Let A be an algebra of the language $\{\vee, \wedge, \to, \bot, 1\}$ with an additional constant 1 for the only distinguished element. A map $v : \{p_0, p_1, \ldots\} \to A$ from the set of propositional variables to the universe of A is called an A-*valuation*. Each A-valuation extends naturally to the set of all formulas. A formula φ is *true* on A, or is an *identity* of A, and we write $\mathsf{A} \models \varphi$, if the equality $v(\varphi) = 1$ holds for any A-valuation v.

Obviously, the set $L\mathsf{A} = \{\varphi \mid \mathsf{A} \models \varphi\}$ of formulas is a logic, which we call a *logic of* A. A *logic of a class of algebras* K is the intersection of logics of algebras in K,

$$L\mathsf{K} = \bigcap\{L\mathsf{A} \mid \mathsf{A} \in \mathsf{K}\}.$$

The algebra A is a *model for* a logic L if $L \subseteq L\mathsf{A}$. If also $L = L\mathsf{A}$, we say that A is a *characteristic model for* L. Every logic in **Jhn** has a characteristic model [11, Ch. III, Sec. 3].

By a j-*algebra* we mean an algebra $\mathsf{A} = \langle A, \wedge, \vee, \to, \bot, 1\rangle$ such that its reduct $\langle A, \wedge, \vee, \to, 1\rangle$ is an implicative lattice and the constant $\bot$ is interpreted as an arbitrary element of the universe A. We recall that $\langle A, \wedge, \vee, \to, 1\rangle$ is an *implicative lattice* if $\langle A, \wedge, \vee, 1\rangle$ is a distributive lattice with the greatest element 1 and $\rightarrow$ is a pseudocomplement operation, i.e., $a \rightarrow b$ is the greatest element in the set $\{c \mid a \wedge c \leqslant b\}$.

A *Heyting algebra* is a j-algebra with the least element $\perp$. A *negative algebra* is a j-algebra with the greatest element $\perp$, i.e., $\perp = 1$.

A *Peirce algebra* is an implicative lattice satisfying the identity $p \vee (p \rightarrow q)$, or, equivalently, $((p \rightarrow q) \rightarrow p) \rightarrow p$. A *Peirce–Johansson algebra* or, shortly, *pj-algebra* (*negative Peirce algebra*, *Boolean algebra*) is a j-algebra (respectively, negative algebra, Heyting algebra) satisfying the identity $p \vee (p \rightarrow q)$.

For any j-algebra A and Heyting algebra B such that they have disjoined universes we will denote by $\mathsf{A} \oplus \mathsf{B}$ a j-algebra obtained by identifying the greatest element of A and the least element of B. Of course, the contradiction of A will play the role of $\perp$ in the resulting algebra $\mathsf{A} \oplus \mathsf{B}$. We denote by $\mathbf{2}$ a two-element Boolean algebra and by $\mathbf{2}'$ a two-element negative algebra.

All classes of algebras defined above form varieties, which define the following logics.

- **Lj** is the logic of the variety of j-algebras.
- **Li** is the logic of the variety of Heyting algebras.
- **Ln** is the logic of the variety of negative j-algebras.
- **Le** is the logic of the variety of pj-algebras.
- **Lk** is the logic of the variety of Boolean algebras.
- **Lmn** is the logic of the variety of negative Peirce algebras.

An element $\star$ of a j-algebra A is said to be the *second greatest element* if for every $x \in \mathsf{A}$, $x \neq 1$ if and only if $x \leqslant \star$. The second greatest element of a j-algebra A will be denoted by $\star_{\mathsf{A}}$, or simply by $\star$.

A j-algebra A is said to be *strongly compact* if there is the second greatest element in A.

It is well known that every strongly compact Boolean algebra is isomorphic to $\mathbf{2}$. Similarly, every strongly compact negative Peirce algebra is isomorphic to $\mathbf{2}'$. In this way, $\mathbf{Lk} = L\mathbf{2}$ and $\mathbf{Lmn} = L\mathbf{2}'$. Every j-algebra have a subalgebra isomorphic to $\mathbf{2}$ or $\mathbf{2}'$. Therefore, **Lj** has exactly two maximal non-trivial extensions.

Proposition 2 [11]. *Every $L \in$ **Jhn** is contained either in **Lk** or in **Lmn**.*

It was proved by McKay [4] that strongly compact Heyting algebras are exactly subdirectly irreducible Heyting algebras. This result can be generalized in a trivial manner to j-algebras.

Proposition 3. *A j-algebra is subdirectly irreducible if and only if it is strongly compact.*

It is well known that a variety of algebras is generated by its finitely generated subdirectly irreducible algebras. Together with previous proposition this yields

Proposition 4. *Let $L \in$ **Jhn** and φ be a formula. We have $\varphi \in L$ if and only if $\mathsf{A} \models \varphi$ for every finitely generated strongly compact j-algebra A modeling L.*

Let us recall Miura's result on the axiomatizability of intersections of logics.

Proposition 5. *Let $L_0 = \mathbf{Lj} + \{\varphi_i \mid i \in I\}$ and $L_1 = \mathbf{Lj} + \{\psi_j \mid j \in J\}$ be finitely axiomatizable extensions of the minimal logic. Then the intersection $L_0 \cap L_1$ is also finitely axiomatizable,*

$$L_0 \cap L_1 = \mathbf{Lj} + \big\{\varphi_i \vee \psi_j^1 \mid i \in I,\ j \in J\big\},$$

where ψ_j^1 is obtained from ψ_j by substitution of propositional variables in such a way that φ_i and ψ_j^1 have no propositional variables in common.

Similar result is well known for extensions of the intuitionistic logic [5]. However, it is not hard to verify that it remains valid for extensions of Johansson's logic.

Corollary 6. $\mathbf{Le} = \mathbf{Lk} \cap \mathbf{Lmn}$.

Indeed, by definition $\mathbf{Lk} = \mathbf{Le} + \{\bot \to p\}$ and $\mathbf{Lmn} = \mathbf{Le} + \{\bot\}$. Therefore, $\mathbf{Lk} \cap \mathbf{Lmn} = \mathbf{Le} + \{\bot \vee (\bot \to p)\} = \mathbf{Le}$ since $\bot \vee (\bot \to p)$ is a substitutional instance of $p \vee (p \to q) \in \mathbf{Le}$.

In conclusion of this section, we prove that the lattice $\mathbf{Jhn}^+$ is distributive. Recall that an *arithmetic variety* is a variety, which is congruence permutable and congruence distributive. According to Pixley's theorem (see [1]) the variety V is arithmetic if and only if there exists a term $m(x, y, z)$ such that the identities

$$m(x, y, x) = m(x, y, y) = m(y, y, x) = x$$

hold in V. In case of j-algebras, as well as in case of Heyting algebras (see [1]), we can use the term

$$m(x, y, z) := \big((x \to y) \to z\big) \wedge \big((z \to y) \to x\big) \wedge (x \vee z)$$

to establish that the varieties of j-algebras and Heyting algebras are arithmetic. The verification is straightforward.

Let us consider ω-generated free j-algebra A_ω and $Con(\mathsf{A}_\omega)$, its congruence lattice. The lattice $Con(\mathsf{A}_\omega)$ is distributive. Moreover, congruences of $Con(\mathsf{A}_\omega)$ are permutable with respect to the composition. Elements of A_ω can be identified with classes of equivalence of formulas with respect to $\mathbf{Lj}$,

$$|\mathsf{A}_\omega| = \big\{[\varphi] \mid \varphi \in \mathsf{F}\big\}.$$

With any $L \in \mathbf{Jhn}^+$ we associate the congruence

$$\theta_L := \big\{([\varphi_0], [\varphi_1]) \mid \varphi_0 \leftrightarrow \varphi_1 \in L\big\}.$$

Clearly, the mapping $L \mapsto \theta_L$ is one-to-one and preserves the ordering. Consequently, to prove that it is a lattice embedding it is enough to check that for any $L_0, L_1 \in \mathbf{Jhn}^+$, the congruences $\theta_{L_0} \wedge \theta_{L_1}$ and $\theta_{L_0} \vee \theta_{L_1}$ also have the form θ_L for suitable L. Observe that θ_L is closed under substitution, i.e., if $[\varphi_0]\theta_L[\varphi_1]$, then $[\varphi_0(\psi_1, \ldots, \psi_n)]\theta_L[\varphi_1((\psi_1, \ldots, \psi_n))]$ for any $\psi_1, \ldots, \psi_n$. It can be easily seen that if $\theta \in Con(\mathsf{A}_\omega)$ is closed under substitution, then $L_\theta = \{\varphi \mid [\varphi]\theta 1\}$, where 1 is the class of **Lj**-tautologies, is a logic from $\mathbf{Jhn}^+$ and $\theta = \theta_{L_\theta}$.

In this way, it is enough to check that $\theta_{L_0} \wedge \theta_{L_1}$ and $\theta_{L_0} \vee \theta_{L_1}$ are closed under substitution. We consider only the non-trivial case of $\theta_{L_0} \vee \theta_{L_1}$. Since A_ω is congruence permutable, $\theta_{L_0} \vee \theta_{L_1} = \theta_{L_0} \circ \theta_{L_1}$. So, $[\varphi_0]\theta_{L_0} \vee \theta_{L_1}[\varphi_1]$ if and only if there is a formula ψ such that $[\varphi_0]\theta_{L_0}[\psi]$ and $[\psi]\theta_{L_1}[\varphi_1]$. This immediately implies that $\theta_{L_0} \vee \theta_{L_1}$ is closed under substitution. We have thus proved

Proposition 7. *The lattice* $\mathbf{Jhn}^+$ *is distributive.*

3. Decomposition of Jhn into three intervals

A natural first step in studying the class of extensions of a logic L is to find its maximal extensions and to classify the logics from this class with respect to their maximal extensions. As was noted in the previous section, the logic **Lj** has two maximal non-trivial extensions, the classical logic $\mathbf{Lk} = L\mathbf{2}$ and the maximal negative logic $\mathbf{Lmn} = L\mathbf{2}'$. So the class **Jhn** is divided into three subclasses. The first subclass consists of all logics admitting only one maximal extension **Lk**, we denote it by **Int**. The second consists of logics contained in **Lmn**, but not in **Lk**. We denote this subclass **Neg**. Finally, the third subclass includes all logics admitting both extensions and we denote this class **Par**. It turns out that each of these subclasses forms an interval in the lattice $\mathbf{Jhn}^+$.

Proposition 8.

(1) $\mathbf{Int} = [\mathbf{Li}, \mathbf{Lk}]$.
(2) $\mathbf{Neg} = [\mathbf{Ln}, \mathbf{Lmn}]$.
(3) $\mathbf{Par} = [\mathbf{Lj}, \mathbf{Le}]$.

Proof. (1) If some logic L is not contained in **Lmn**, then any model of L is a Heyting algebra. Indeed, assume that $\mathsf{A} \models L$ and there is $a \in A$ strictly less than $\perp$. Then the quotient algebra $\mathsf{A}/\langle\perp\rangle$, where $\langle\perp\rangle$ denotes the principal filter generated by the element $\perp$, is non-trivial. Take $b \in \mathsf{A}/\langle\perp\rangle$ such that $b \neq \perp$. Then $\mathsf{A}/\langle\perp\rangle$ has a subalgebra with the universe $\{\perp, b\}$, which is isomorphic to $\mathbf{2}'$. In this way, $L \subseteq \mathbf{Lmn}$, a contradiction. Thus, $\perp$ is the least element in all models of L and the formula $\perp \to p$ belongs to L, i.e., $\mathbf{Li} \subseteq L$.

(2) Assume L is not contained in **Lk** and prove that all models of L are negative. If $\mathsf{A} \models L$ and $1 \neq \perp$, then A has a subalgebra with the universe $\{1, \perp\}$, which is isomorphic to $\mathbf{2}$, i.e., $L \subseteq \mathbf{Lk}$, a contradiction. Thus, $\perp = 1$ in all models of L and $\mathbf{Ln} = \mathbf{Lj} + \{\perp\} \subseteq L$.

(3) If L has two maximal extensions, then it is contained in $\mathbf{Lk} \cap \mathbf{Lmn}$, which is equal to **Le** by Corollary 6. □

Note that all logics of **Int** are explosive and all logics of **Neg** have a degenerate negation in the sense that any negated formula is provable in them. Moreover, these classes are well studied; logics of **Neg** are definitionally equivalent to positive logics. So the class **Par** contains all interesting cases of paraconsistent negations in the class **Jhn**. We start the investigation of this class by defining for its logics counterparts in classes **Int** and **Neg**.

4. Intuitionistic and negative counterparts

We define two mappings $(-)_{int} : \mathbf{Jhn}^{+} \to \mathbf{Int}$ and $(-)_{neg} : \mathbf{Jhn}^{+} \to \mathbf{Neg}$ as follows. For any $L \in \mathbf{Jhn}^{+}$, put $L_{int} := L + \{\bot \to p\} = L \vee \mathbf{Li}$ and $L_{neg} := L + \{\bot\} = L \vee \mathbf{Ln}$.

Proposition 9.

(1) *The mappings $(-)_{int}$ and $(-)_{neg}$ are lattice epimorphisms.*
(2) *For any $L \in \mathbf{Jhn}^{+}$, $L_{int} = \mathsf{F}$ if and only if $L \in \mathbf{Neg}$.*
(3) *For any $L \in \mathbf{Jhn}^{+}$, $L_{neg} = \mathsf{F}$ if and only if $L \in \mathbf{Int}$.*

Proof. (1) This fact easily follows from the distributivity of $\mathbf{Jhn}^{+}$.

(2) If $L \in \mathbf{Neg}$, then $\bot, \bot \to p \in L_{int}$, which leads to explosion. If L belongs to $\mathbf{Par} \cup \mathbf{Int}$, then L, as well as $\mathbf{Li}$ is contained in $\mathbf{Lk}$, consequently L_{int} is also contained in $\mathbf{Lk}$.

(3) If $L \in \mathbf{Int}$, then $\bot, \bot \to p \in L_{neg}$, which leads to explosion. If L is in $\mathbf{Par} \cup \mathbf{Neg}$, then L, as well as $\mathbf{Ln}$ is contained in $\mathbf{Lmn}$, consequently L_{neg} is also contained in $\mathbf{Lmn}$. $\square$

The following statement gives an alternative definition of counterparts. Let $In(\varphi(p_1, \ldots, p_n)) := \varphi(p_1 \vee \bot, \ldots, p_n \vee \bot)$ for any formula φ with propositional variables among $p_1, \ldots, p_n$.

Proposition 10. *Let $L \in \mathbf{Par}$. Then*

$$L_{int} = \{\varphi \mid In(\varphi) \in L\} \quad \textit{and} \quad L_{neg} = \{\varphi \mid \bot \to \varphi \in L\}.$$

Proof. Let $L^* := \{\varphi \mid In(\varphi) \in L\}$. We check first that L^* is a logic.

Lemma 11. *For any φ, $\mathbf{Lj} \vdash (In(\varphi) \vee \bot) \leftrightarrow In(\varphi)$.*

Proof. We establish by a trivial induction on the structure of formulas that $\mathbf{Lj} \vdash \bot \to In(\varphi)$ for any φ. The desired conclusion can be easily deduced from this fact. $\square$

If $\varphi(p_1, \ldots, p_n) \in L^*$, then the formula $\varphi(p_1 \vee \bot, \ldots, p_n \vee \bot)$ and its substitutional instance $\varphi(In(\psi_1) \vee \bot, \ldots, In(\psi_n) \vee \bot)$ are in L. By Lemma 11 we have $(In(\psi_i) \vee \bot) \leftrightarrow In(\psi_i) \in \mathbf{Lj}$, $i = 1, \ldots, n$, which yields $\varphi(In(\psi_1), \ldots, In(\psi_n)) = In(\varphi(\psi_1, \ldots, \psi_n)) \in L$, i.e., $\varphi(\psi_1, \ldots, \psi_n) \in L^*$.

If $\varphi, \varphi \to \psi \in L^*$, then $In(\varphi)$, $In(\varphi \to \psi) = In(\varphi) \to In(\psi) \in L$, whence $I(\psi) \in L$ and $\psi \in L^*$. Thus, L^* is closed under substitution and *modus ponens*.

The fact that $In(\varphi)$ was defined as a substitutional instance of φ implies $L \subseteq L^*$. At the same time, $In(\bot \to p) = \bot \to (p \vee \bot) \in \mathbf{Lj} \subseteq L$. Thus, $L_{int} \subseteq L^*$. To check the inverse inclusion, take a $\varphi = \varphi(p_1, \ldots, p_n) \in L^*$. Then $In(\varphi) \in L \subseteq L_{int}$. Equivalences $p_i \leftrightarrow (p_i \vee \bot)$ hold in L_{int}, which allows one to conclude $L_{int} \vdash In(\varphi) \leftrightarrow \varphi$, whence $\varphi \in L_{int}$.

The first equality is thus proved, the second one immediately follows from the deduction theorem. $\square$

Define a *contradiction operator* as $C(\varphi) := \varphi \wedge \neg\varphi$ and extend this definition to sets of formulas as follows: $C(\emptyset) := \{\bot\}$ and $C(X) := \{C(\varphi) \mid \varphi \in X\}$ for non-empty X. This operator defines a strong translation of the negative counterpart into a paraconsistent logic. More exactly, the following holds.

Proposition 12. *Let $L \in \mathbf{Par}$, X be a set of formulas, and φ a formula. Then $X \vdash_{L_{neg}} \varphi$ if and only if $C(X) \vdash_L C(\varphi)$.*

Proof. Since $L_{neg} = L + \{\bot\}$, the relation $X \vdash_{L_{neg}} \varphi$ holds if and only if $X \cup \{\bot\} \vdash_L \varphi$ or, equivalently, $X \cup \{\bot\} \vdash_L \varphi \wedge \bot$. Taking into account that $(\varphi \wedge \neg\varphi) \equiv (\varphi \wedge \bot) \in \mathbf{Lj}$ one can easily obtain that the last consequence relation is equivalent to $C(X) \vdash_L C(\varphi)$. □

This statement shows that the counterpart L_{neg} explicate the non-trivial structure of contradictions of the paraconsistent logic L. The formulas of L_{neg} behave itself exactly so as constructed from them contradictions in L.

We have thus defined for any logic from **Par** its counterparts in the classes **Int** and **Neg** and shown that these counterparts can be translated into the original logic. Turn now to the inverse problem. Given logics $I \in \mathbf{Int}$ and $N \in \mathbf{Neg}$, which logics from **Par** have I and N as intuitionistic and negative counterparts, respectively? In other words, let us consider the families of logics

$$Spec(I, N) := \{L \in \mathbf{Par} \mid L_{int} = I, L_{neg} = N\}$$

for $I \in \mathbf{Int}$ and $N \in \mathbf{Neg}$. All families of this form are non-empty and form intervals in the lattice $\mathbf{Jhn}^+$.

Proposition 13. *Let $I \in \mathbf{Int}$ and $N \in \mathbf{Neg}$. Then*

$$Spec(I, N) = [I * N, I \cap N],$$

where $I * N = \mathbf{Lj} + \{In(\varphi), \bot \to \psi \mid \varphi \in I, \psi \in N\}$.

Proof. In view of Proposition 9, if $L_1 \subseteq L_2$, then $(L_1)_{int} \subseteq (L_2)_{int}$ and $(L_1)_{neg} \subseteq (L_2)_{neg}$. This fact implies that the set $Spec(I, N)$ is convex with respect to the lattice order of $\mathbf{Jhn}^+$.

By definition, $L \subseteq L_{int}, L_{neg}$ for any L. Therefore, for any $L \in Spec(I, N)$, we have $L \subseteq I \cap N$. On the other hand, if $L \in Spec(I, N)$, then

$$\{In(\varphi), \bot \to \psi \mid \varphi \in I,\ \psi \in N\} \subseteq L,$$

i.e., $I * N \subseteq L$.

It remains to check that the logics $I \cap N$ and $I * N$ are in $Spec(I, N)$. Clearly, $\{In(\varphi) \mid \varphi \in I\} \subseteq I$ and $\{\bot \to \psi \mid \psi \in N\} \subseteq N$. At the same time, any formula $\bot \to \psi$ belongs to I since $I \in \mathbf{Int}$, and any formula $In(\varphi)$ belongs to the negative logic N, which can be stated by a trivial induction on the structure of formulas. In this way, $I * N \subseteq I \cap N$, which implies by Proposition 9 $(I * N)_{int} \subseteq (I \cap N)_{int}$ and $(I * N)_{neg} \subseteq (I \cap N)_{neg}$.

By definition of $I * N$, we have

$$I \subseteq (I * N)_{int} \quad \text{and} \quad N \subseteq (I * N)_{neg}.$$

At the same time,

$$(I \cap N)_{int} \subseteq I_{int} = I \quad \text{and} \quad (I \cap N)_{neg} \subseteq N_{neg} = N.$$

Combining all these facts we obtain $I * N, I \cap N \in \mathit{Spec}(I, N)$. □

It is interesting that the upper points of intervals of the form $\mathit{Spec}(I, N)$ also form an interval in the lattice $\mathbf{Jhn}^+$. We put $\mathbf{Le}' := \mathbf{Lj} + \{\bot \vee (\bot \to p)\}$.

Proposition 14. *The mapping* $(I, N) \mapsto I \cap N$ *defines a lattice isomorphism of the direct product* $\mathbf{Int} \times \mathbf{Neg}$ *onto the interval* $[\mathbf{Le}', \mathbf{Le}]$. *The inverse isomorphism is given by the rule* $L \mapsto (L_{int}, L_{neg})$.

Proof. If $L = I \cap N$ for some $I \in \mathbf{Int}$ and $N \in \mathbf{Neg}$, then

$$I = L_{int} = L + \{\bot \to p\} \quad \text{and} \quad N = L_{neg} = L + \{\bot\}$$

as it was established in the proof of the previous proposition. Taking into account Proposition 5 we obtain $L = L + \{\bot \vee (\bot \to p)\}$, i.e., $\mathbf{Le}' \subseteq L$. It is clear that $L \in \mathbf{Par}$ as an intersection of intermediate and negative logics, therefore, $L \subseteq \mathbf{Le}$ by Proposition 8.

Conversely, let $L \in [\mathbf{Le}', \mathbf{Le}]$. We have then

$$L_{int} \cap L_{neg} = \big(L + \{\bot \to p\}\big) \cap \big(L + \{\bot\}\big) = L + \big\{\bot \vee (\bot \to p)\big\} = L,$$

which means that L can be represented as an intersection of intermediate and negative logics. We have thus proved that $(I, N) \mapsto I \cap N$ is a mapping of $\mathbf{Int} \times \mathbf{Neg}$ onto $[\mathbf{Le}', \mathbf{Le}]$.

The equations $(I \cap N)_{int} = I$ and $(I \cap N)_{neg} = N$ imply that this mapping is one-to-one. The fact that this mapping is homomorphic follows immediately from the definition. □

5. Three dimensions of Par

We can see now that the class **Par** has a three-dimensional structure. The position of a logic L in this class is determined by its intuitionistic counterpart L_{int}, which can be considered as a logic modeling the reasoning in L under the additional assumption of inconsistency, or of impossibility of contradictions, and by its structure of contradictions explicated in the negative counterpart L_{neg}. When inconsistent patterns of reasoning and the structure of contradictions are fixed, we have the further variety of possibilities for combining them presented by the interval of logics $\mathit{Spec}(L_{int}, L_{neg})$. The place of L in this interval can be considered as its third coordinate in **Par**, the sense of which is not quite clear yet. It becomes clearer in the next section. Now we turn to the fact that unlike the first and second coordinates having absolute scales, **Int** and **Neg**, respectively, the size of the scale for the third coordinate is dependent of the first two coordinates. However, one can find natural interrelations between these scales, i.e., between the intervals of the form $\mathit{Spec}(I, N)$ for different $I \in \mathbf{Int}$ and $N \in \mathbf{Neg}$.

Consider two pairs of logics $\mathsf{P}_1 = (I_1, N_1)$ and $\mathsf{P}_2 = (I_2, N_2)$, where $I_1, I_2 \in \mathbf{Int}$, $N_1, N_2 \in \mathbf{Neg}$. The relation $\mathsf{P}_1 \leqslant \mathsf{P}_2$ means that $I_1 \subseteq I_2$ and $N_1 \subseteq N_2$. We will write also $\mathit{Spec}(\mathsf{P}_1)$ for $\mathit{Spec}(I_1, N_1)$.

Let $\mathsf{P}_1 = (I_1, N_1)$ and $\mathsf{P}_2 = (I_2, N_2)$ be such that $\mathsf{P}_1 \leqslant \mathsf{P}_2$. Mappings $r_{\mathsf{P}_2,\mathsf{P}_1} : Spec(\mathsf{P}_2) \to$ **Par** and $e_{\mathsf{P}_1,\mathsf{P}_2} : Spec(\mathsf{P}_1) \to$ **Par** are defined as follows

$$r_{\mathsf{P}_2,\mathsf{P}_1}(L) := L \cap (I_1 \cap N_1), \qquad e_{\mathsf{P}_1,\mathsf{P}_2}(L) := L \vee (I_2 * N_2).$$

Proposition 15. *Let the pairs of logics $\mathsf{P}_1, \mathsf{P}_2$ be such that $\mathsf{P}_1 \leqslant \mathsf{P}_2$. The following facts hold.*

(1) *For any $L \in Spec(\mathsf{P}_2)$, we have $e_{\mathsf{P}_1,\mathsf{P}_2} r_{\mathsf{P}_2,\mathsf{P}_1}(L) = L$.*
(2) *For any $L \in Spec(\mathsf{P}_1)$, we have*

$$r_{\mathsf{P}_2,\mathsf{P}_1} e_{\mathsf{P}_1,\mathsf{P}_2}(L) = L \vee r_{\mathsf{P}_2,\mathsf{P}_1}(I_2 * N_2).$$

(3) *$e_{\mathsf{P}_1,\mathsf{P}_2}$ is a lattice epimorphism from $Spec(\mathsf{P}_1)$ onto $Spec(\mathsf{P}_2)$.*
(4) *$r_{\mathsf{P}_2,\mathsf{P}_1}$ is a lattice monomorphism from $Spec(\mathsf{P}_2)$ into $Spec(\mathsf{P}_1)$ and has the following image*

$$r_{\mathsf{P}_2,\mathsf{P}_1}(\mathsf{P}_2) = \left[r_{\mathsf{P}_2,\mathsf{P}_1}(I_2 * N_2), I_1 \cap N_1\right].$$

(5) *For any P_3 such that $\mathsf{P}_2 \leqslant \mathsf{P}_3$, we have*

$$e_{\mathsf{P}_1,\mathsf{P}_2} e_{\mathsf{P}_2,\mathsf{P}_3} = e_{\mathsf{P}_1,\mathsf{P}_3}, \qquad r_{\mathsf{P}_3,\mathsf{P}_2} r_{\mathsf{P}_2,\mathsf{P}_1} = r_{\mathsf{P}_3,\mathsf{P}_1}.$$

Proof. (1) We calculate

$$\begin{aligned} e_{\mathsf{P}_1,\mathsf{P}_2} r_{\mathsf{P}_2,\mathsf{P}_1}(L) &= \big(L \cap (I_1 \cap N_1)\big) \vee (I_2 * N_2) \\ &= \big(L \vee (I_2 * N_2)\big) \cap \big((I_1 \cap N_1) \vee (I_2 * N_2)\big). \end{aligned}$$

By Proposition 13, $I_2 * N_2$ is the least point of $Spec(\mathsf{P}_2)$, therefore, we have $L \vee (I_2 * N_2) = L$. Further, we need one lemma.

Lemma 16. *For any $L \in Spec(I, N)$, $I \cap N = L \vee \mathbf{Le}'$.*

Proof. $(\mathbf{Le}')_{int}$ equals to **Li**, the least logic in **Int**, and $(\mathbf{Le}')_{neg} = \mathbf{Ln}$, which is the least logic in **Neg**. Now, it follows from Proposition 9 that $L \vee \mathbf{Le}'$ has the same counterparts as L. By Proposition 14, $L \vee \mathbf{Le}'$ coincides with the greatest point of $Spec(I, N)$. □

Using this lemma and the obvious relation $I_1 * N_1 \subseteq I_2 * N_2$ we obtain

$$(I_1 \cap N_1) \vee (I_2 * N_2) = \big((I_1 * N_1) \vee \mathbf{Le}'\big) \vee (I_2 * N_2) = I_2 * N_2 \vee \mathbf{Le}' = I_2 \cap N_2.$$

And finally, $e_{\mathsf{P}_1,\mathsf{P}_2} r_{\mathsf{P}_2,\mathsf{P}_1}(L) = L \cap (I_2 \cap N_2) = L$.

(2) The direct computation shows

$$\begin{aligned} r_{\mathsf{P}_2,\mathsf{P}_1} e_{\mathsf{P}_1,\mathsf{P}_2}(L) &= \big(L \vee (I_2 * N_2)\big) \cap (I_1 \cap N_1) \\ &= \big(L \cap (I_1 \cap N_1)\big) \vee \big((I_2 * N_2) \cap (I_1 \cap N_1)\big) = L \vee r_{\mathsf{P}_2,\mathsf{P}_1}(I_2 * N_2). \end{aligned}$$

(3) It follows from the definition and the distributivity of $\mathbf{Jhn}^+$ that $e_{\mathsf{P}_1,\mathsf{P}_2}$ is a lattice homomorphism. Let $L \in Spec(\mathsf{P}_1)$ and $L' := e_{\mathsf{P}_1,\mathsf{P}_2}(L) = L \vee (I_2 * N_2)$. By Proposition 9,

$(L')_{int} = L_{int} \vee (I_2 * N_2)_{int} = I_1 \vee I_2 = I_2$. In the same way, $(L')_{neg} = N_2$, consequently, $L' \in Spec(\mathsf{P}_2)$. The fact that $e_{\mathsf{P}_1,\mathsf{P}_2}$ is an epimorphism will follow from item (1).

(4) As above, we use Proposition 9 to check that $r_{\mathsf{P}_2,\mathsf{P}_1}$ maps $Spec(\mathsf{P}_2)$ into $Spec(\mathsf{P}_1)$. This is a homomorphism due to the distributivity of $\mathbf{Jhn}^+$. Applying the formula of item (1) to $r_{\mathsf{P}_2,\mathsf{P}_1}(L_1) = r_{\mathsf{P}_2,\mathsf{P}_1}(L_2)$ we obtain $L_1 = L_2$. Thus, $r_{\mathsf{P}_2,\mathsf{P}_1}$ is a monomorphism. The fact that $r_{\mathsf{P}_2,\mathsf{P}_1}$ preserves the ordering of **Jhn** implies the inclusion $r_{\mathsf{P}_2,\mathsf{P}_1}(\mathsf{P}_2) \subseteq [r_{\mathsf{P}_2,\mathsf{P}_1}(I_2 * N_2), I_1 \cap N_1]$. To check the inverse inclusion take a logic $L \in Spec(\mathsf{P}_1)$ with the property $r_{\mathsf{P}_2,\mathsf{P}_1}(I_2 * N_2) \subseteq L$. Then by item (2)

$$r_{\mathsf{P}_2,\mathsf{P}_1} e_{\mathsf{P}_1,\mathsf{P}_2}(L) = L \vee r_{\mathsf{P}_2,\mathsf{P}_1}(I_2 * N_2) = L,$$

i.e., $L \in r_{\mathsf{P}_2,\mathsf{P}_1}(\mathsf{P}_2)$.

(5) This item follows from the obvious relations $I_2 * N_2 \subseteq I_3 * N_3$ and $I_1 \cap N_1 \subseteq I_2 \cap N_2$. $\square$

The above proposition shows, in particular, that any interval $Spec(I, N)$ is isomorphic to an upper subinterval of $Spec(\mathbf{Li}, \mathbf{Ln})$. In this way, this latter interval can be considered as a scale for the third dimension of the class **Par**. Extending the intuitionistic and negative counterparts, we restrict simultaneously the part of the scale, which can be used to construct a logic with the given counterparts. It is worth noticing also the following consequence of the last proposition.

Corollary 17. *Let $\mathsf{P}_1 = (I_1, N_1)$ and $\mathsf{P}_2 = (I_2, N_2)$ be pairs of logics such that $\mathsf{P}_1 \leqslant \mathsf{P}_2$. For any logics $L_1, L_2 \in Spec(\mathsf{P}_2)$, $L_1 \neq L_2$, there is a formula $\varphi \in I_1 \cap N_1$ such that $\varphi \in L_1 \triangle L_2$, where $\triangle$ denotes a symmetrical difference of sets.*

Proof. Let $L_1, L_2 \in Spec(\mathsf{P}_2)$. If $L_1 \neq L_2$, but these logics are not distinguished by a formula $\varphi \in I_1 \cap N_1$, then $r_{\mathsf{P}_2,\mathsf{P}_1}(L_1) = r_{\mathsf{P}_2,\mathsf{P}_1}(L_2)$. By item (2) of the previous proposition, $r_{\mathsf{P}_2,\mathsf{P}_1}$ is a monomorphism, whence, $L_1 = L_2$, a contradiction. $\square$

In particular, any two logics from an interval $Spec(I, N)$ can be distinguished via a formula from $\mathbf{Le}' = \mathbf{Li} \cap \mathbf{Ln}$. Moreover, any logic from the interval $Spec(I, N)$ can be axiomatized by formulas from $\mathbf{Le}'$ modulo the least logic of the interval $I * N$. Indeed, for any $L \in Spec(I, N)$ we have by Proposition 15(3) $L = (L \cap \mathbf{Le}') \vee (I * N)$.

For further investigations of the structure of the class **Par**, we need semantical considerations.

6. Representation of j-algebras

Let A be a j-algebra. We put

$$A^{\perp} := \{a \in A \mid a \geqslant \perp\} \quad \text{and} \quad A_{\perp} := \{a \in A \mid a \leqslant \perp\}.$$

The set $A^{\perp}$ is obviously closed under the operations of A and $\perp$ is the least element of $A^{\perp}$. In this way, we can define a Heyting algebra $\mathsf{A}^{\perp}$ as a subalgebra of A with the universe $A^{\perp}$. The set $A_{\perp}$ is a sublattice of A, but it is not a subalgebra of A, except for the case $\perp = 1$.

However, the operation

$$x \to_\perp y := (x \to y) \wedge \perp$$

turns $A_\perp$ into a j-algebra, which we denote $\mathsf{A}_\perp$. Obviously, $\mathsf{A}_\perp$ is a negative algebra. We call $\mathsf{A}^\perp$ and $\mathsf{A}_\perp$ *upper* and *lower algebras* of A, respectively. The notions of upper and lower algebras are semantical analogues of intuitionistic and negative counterparts.

Proposition 18. *Let* A *be a* j*-algebra and* φ *a formula. The following equivalences take place*:

$$\mathsf{A}^\perp \models \varphi \ \Leftrightarrow\ \mathsf{A} \models \mathit{In}(\varphi), \qquad \mathsf{A}_\perp \models \varphi \ \Leftrightarrow\ \mathsf{A} \models \perp \to \varphi.$$

Proof. Check the first equivalence. Assume $\mathsf{A}^\perp \models \varphi$ and prove $\mathsf{A} \models \mathit{In}(\varphi)$. For an A-valuation v define an $\mathsf{A}^\perp$-valuation v' by the rule $v'(p) := v(p) \vee \perp$. It follows easily that $v(\mathit{In}(\varphi)) = v'(\varphi)$, which immediately implies the desired conclusion.

Conversely, let $\mathsf{A} \models \mathit{In}(\varphi)$. For any $\mathsf{A}^\perp$-valuation v, we have $v = v'$, in particular, $v(\mathit{In}(\varphi)) = v(\varphi)$, which completes the proof of the equivalence.

To prove the second equivalence we need the following

Lemma 19. *For any* j*-algebra* A*, the mapping* $\tau : \mathsf{A} \to \mathsf{A}_\perp$, $\tau(x) = x \wedge \perp$*, is an epimorphism of* j*-algebras.*

The proof immediately follows from the definition of implication in $\mathsf{A}_\perp$ and the identity $(x \to y) \wedge z = ((x \wedge z) \to (y \wedge z)) \wedge z$ satisfied in all j-algebras.

Assuming $\mathsf{A}_\perp \models \varphi$ we take an A-valuation v and consider the composition τv, which is an $\mathsf{A}_\perp$-valuation. In view of the fact that τ is an epimorphism, $v(\varphi \wedge \perp) = \tau v(\varphi)$. But $\tau v(\varphi) = \perp$ by assumption, which yields the equalities $v(\perp \to (\varphi \wedge \perp)) = v(\perp \to \varphi) = 1$. Thus, $\mathsf{A} \models \perp \to \varphi$.

Now, we let $\mathsf{A} \models \perp \to \varphi$. Let v be an $\mathsf{A}_\perp$-valuation and v' be an A-valuation such that $v(p) = v'(p)$ for all p. Clearly, $v = \tau v'$. By assumption we have $\perp \leqslant v'(\varphi \wedge \perp) = \tau v'(\varphi) = v(\varphi)$, $\perp$ is the greatest element of $\mathsf{A}_\perp$, whence, $v(\varphi) = \perp$. In this way, $\mathsf{A}_\perp \models \varphi$. $\square$

It follows from this fact that for a j-algebra A modeling L, its upper and lower algebras model intuitionistic and negative counterparts of L, respectively.

Corollary 20. *Let* A *be a* j*-algebra,* $L \in$ **Par**, $I \in$ **Int**, *and* $N \in$ **Neg**. *The following facts are true.*

(1) *If* $\mathsf{A} \models L$*, then* $\mathsf{A}^\perp \models L_{int}$ *and* $\mathsf{A}_\perp \models L_{neg}$.
(2) $\mathsf{A} \models I * N$ *if and only if* $\mathsf{A}^\perp \models I$ *and* $\mathsf{A}_\perp \models N$.

Proof. (1) This is a direct consequence of the previous proposition and the definition of counterparts.

(2) If $\mathsf{A} \models I * N$, then $\mathsf{A}^{\perp} \models I$ and $\mathsf{A}_{\perp} \models N$ by item (1). If $\mathsf{A}^{\perp} \models I$ and $\mathsf{A}_{\perp} \models N$, then by Proposition 18, $\mathsf{A} \models \{In(\varphi), \perp \to \psi \colon \varphi \in I, \ \psi \in N\}$. The latter means exactly that $\mathsf{A} \models I * N$. $\square$

For a class K of j-algebras, let

$$\mathsf{K}^{\perp} := \{\mathsf{A}^{\perp} \mid \mathsf{A} \in \mathsf{K}\}, \qquad \mathsf{K}_{\perp} := \{\mathsf{A}_{\perp} \mid \mathsf{A} \in \mathsf{K}\}.$$

The following proposition is a generalization of Corollary 20(1) to classes of algebras.

Proposition 21. *Let K be a class of j-algebras. Then*

$$(L\mathsf{K})_{int} = L\mathsf{K}^{\perp}, \quad \text{and} \quad (L\mathsf{K})_{neg} = L\mathsf{K}_{\perp}.$$

Proof. A formula φ belongs to $(L\mathsf{K})_{int}$ if and only if $In(\varphi) \in L\mathsf{K}$. Due to Proposition 18 the latter means that for any $\mathsf{A} \in \mathsf{K}$, $\mathsf{A}^{\perp} \models \varphi$, i.e., $\varphi \in L\mathsf{K}^{\perp}$. The second equality can be proved similarly. $\square$

It is interesting that the second statement of Corollary 20 cannot be generalized in a similar way to classes of algebras. If a class K_1 of Heyting algebras defines a logic $I \in \mathbf{Int}$ and a class K_2 of negative algebras defines a logic $N \in \mathbf{Neg}$, then the class of all algebras with the upper algebra in K_1 and the lower algebra in K_2, i.e., the class

$$\mathsf{K}_1 * \mathsf{K}_2 := \{\mathsf{A} \mid \mathsf{A}^{\perp} \in \mathsf{K}_1, \ \mathsf{A}_{\perp} \in \mathsf{K}_2\},$$

does not determine, in general case, the logic $I * N$. For example, $\mathbf{Lk} = L\mathbf{2}$, $\mathbf{Lmn} = L\mathbf{2}'$, but

$$L\{\mathbf{2}\} * \{\mathbf{2}'\} = L\mathbf{3} \neq \mathbf{Lk} * \mathbf{Lmn},$$

where $\mathbf{3}$ is a three-element chain with $\perp$ interpreted as an intermediate element (see Proposition 30 below).

The above remarks give rise to a question about the structure of an arbitrary j-algebra with given upper and lower algebras. First, we give a construction of j-algebras, which have given Heyting and negative algebras as their upper and lower algebras respectively. Let B be a Heyting algebra, C a negative algebra, and $f : C \to B$ a lower semilattice homomorphism preserving the unit element. Let

$$B \times_f C := \big\{(b, c) \mid b \in B, c \in C, b \leqslant f(c)\big\}.$$

Proposition 22. *The set $B \times_f C$ is closed under the componentwise lattice operations $\vee$ and $\wedge$ and under the implication $\to$ defined as follows*

$$(b_1, c_1) \to (b_2, c_2) := \big((b_1 \to_{\mathsf{B}} b_2) \wedge f(c_1 \to_{\mathsf{C}} c_2), c_1 \to_{\mathsf{C}} c_2\big).$$

The algebraic system $\mathsf{B} \times_f \mathsf{C} := \langle B \times_f C, \vee, \wedge, \to, \perp, 1\rangle$, where $\perp := (\perp_{\mathsf{B}}, \perp_{\mathsf{C}})$ and $1 := (1_{\mathsf{B}}, \perp_{\mathsf{C}})$, is a j-algebra. Moreover, $(\mathsf{B} \times_f \mathsf{C})^{\perp} \cong \mathsf{B}$ and $(\mathsf{B} \times_f \mathsf{C})_{\perp} \cong \mathsf{C}$.

Proof. We check that the pseudocomplement operation is well defined. Let $b_1, b_2 \in B$, $c_1, c_2 \in C$, $b_1 \leqslant f(c_1)$, and $b_2 \leqslant f(c_2)$. The element $(b_1, c_1) \to (b_2, c_2)$, if it is defined, must be greatest among the elements (x, y) such that $x \leqslant f(y)$ and $(b_1, c_1) \wedge (x, y) \leqslant (b_2, c_2)$. The latter is equivalent to relations $x \leqslant (b_1 \to b_2) \wedge f(y)$ and $y \leqslant c_1 \to c_2$. Taking into account that f preserves the ordering we immediately obtain that the desired pseudocomplement is equal to $((b_1 \to b_2) \wedge f(c_1 \to c_2),\ c_1 \to c_2)$. Thus, $\mathsf{B} \times_f \mathsf{C}$ is a j-algebra. Its upper and lower algebras have the form

$$\left|(\mathsf{B} \times_f \mathsf{C})^{\perp}\right| = \{(b, \perp) \mid b \in B\} \quad \text{and} \quad \left|(\mathsf{B} \times_f \mathsf{C})_{\perp}\right| = \{(\perp, c) \mid c \in C\}.$$

It is clear that the mappings $(b, \perp) \mapsto b$, $b \in B$, and $(\perp, c) \mapsto c$, $c \in C$, determine isomorphisms of $(\mathsf{B} \times_f \mathsf{C})^{\perp}$ and B and of $(\mathsf{B} \times_f \mathsf{C})_{\perp}$ and C, respectively. □

Every j-algebra can be presented in the above form.

Proposition 23. *For any j-algebra A, the mapping $f_{\mathsf{A}} : \mathsf{A}_{\perp} \to \mathsf{A}^{\perp}$ given by the rule $f_{\mathsf{A}}(x) := \perp \vee (\perp \to x)$ is a lower semilattice homomorphism preserving the unit element, and the mapping $\lambda(x) := (x \vee \perp, x \wedge \perp)$ defines an isomorphism*

$$\mathsf{A} \cong \mathsf{A}^{\perp} \times_{f_{\mathsf{A}}} \mathsf{A}_{\perp}.$$

Proof. We verify that f_{A} is a semilattice homomorphism preserving the unit element. For brevity, we omit the lower index in the denotation f_{A}. We have $f(\perp) = \perp \vee (\perp \to \perp) = 1$. Further,

$$\begin{aligned} f(y_1) \wedge f(y_2) &= \bigl(\perp \vee (\perp \to y_1)\bigr) \wedge \bigl(\perp \vee (\perp \to y_2)\bigr) \\ &= \perp \vee \bigl((\perp \to y_1) \wedge (\perp \to y_2)\bigr) = \perp \vee \bigl(\perp \to (y_1 \wedge y_2)\bigr) \\ &= f(y_1 \wedge y_2). \end{aligned}$$

Thus, the j-algebra $\mathsf{B} := \mathsf{A}^{\perp} \times_{f_{\mathsf{A}}} \mathsf{A}_{\perp}$ is well defined.

Let us check that λ maps A onto B. In **Lj**, one can easily prove the formula $(p \vee \perp) \to (\perp \vee (\perp \to (p \wedge \perp)))$. This means that for any $a \in \mathsf{A}$, the inequality $a \vee \perp \leqslant \perp \vee (\perp \to (a \wedge \perp))$ holds, i.e., $\lambda(a) \in \mathsf{B}$. Now, let $a, b \in \mathsf{A}$, $a \geqslant \perp$, $b \leqslant \perp$, and $a \leqslant \perp \vee (\perp \to b)$. We show that there exists an element $c \in \mathsf{A}$ such that $a = c \vee \perp$ and $b = c \wedge \perp$. Put $c = a \wedge (\perp \to b)$, then $c \vee \perp = (\perp \vee a) \wedge (\perp \vee (\perp \to b)) = a \wedge (\perp \vee (\perp \to b)) = a$ and also $c \wedge \perp = a \wedge (\perp \to b) \wedge \perp = \perp \wedge (\perp \to b) = b$. Thus, the mapping $\lambda : \mathsf{A} \to \mathsf{B}$ is onto. It is the well-known fact from the lattice theory that the mapping of the form $x \mapsto (x \vee \perp, x \wedge \perp)$ defines a lattice embedding of A into the direct product $\mathsf{A}^{\perp} \times \mathsf{A}_{\perp}$. We have thus proved that λ is a lattice isomorphism of A and $\mathsf{A}^{\perp} \times_{f_{\mathsf{A}}} \mathsf{A}_{\perp}$ and it remains to check that λ preserves the pseudocomplement operation.

Let $a, b \in A$. We have $a \wedge (a \to b) \leqslant b$. The lattice isomorphism properties of λ imply $\lambda(a) \wedge \lambda(a \to b) \leqslant \lambda(b)$, therefore, $\lambda(a \to b) \leqslant \lambda(a) \to \lambda(b)$.

Due to the fact that λ is onto we have $\lambda(a) \to \lambda(b) = \lambda(c)$ for some $c \in A$. The relation $\lambda(a) \wedge \lambda(c) \leqslant \lambda(b)$ implies again $a \wedge c \leqslant b$, i.e., $c \leqslant a \to b$, and finally, $\lambda(c) = \lambda(a) \to \lambda(b) \leqslant \lambda(a \to b)$. □

Corollary 24. *Let $I \in \mathbf{Int}$ and $N \in \mathbf{Neg}$. A j-algebra A models $I \cap N$ if and only if $\mathsf{A} \cong \mathsf{A}^{\perp} \times \mathsf{A}_{\perp}$, $\mathsf{A}^{\perp} \models I$, and $\mathsf{A}_{\perp} \models N$.*

Proof. If $\mathsf{A} \models I \cap N$, then by Corollary 20 $\mathsf{A}^{\perp} \models I = (I \cap N)_{int}$ and $\mathsf{A}_{\perp} \models N = (I \cap N)_{neg}$. Moreover, in this case, $\mathsf{A} \models \perp \vee (\perp \to p)$ (see Proposition 14), whence $f_{\mathsf{A}}(x) = 1$ for any $x \in A$. The latter fact means that the pseudocomplement operation of $\mathsf{A}^{\perp} \times_{f_{\mathsf{A}}} \mathsf{A}_{\perp}$ is componentwise and that $a \leqslant f(b)$ for any $a \in \mathsf{A}^{\perp}$ and $b \in \mathsf{A}_{\perp}$. In this way, $\mathsf{A}^{\perp} \times_{f_{\mathsf{A}}} \mathsf{A}_{\perp}$ coincides with the direct product of $\mathsf{A}^{\perp} \times \mathsf{A}_{\perp}$.

The inverse implication is obvious. □

The presentation of j-algebras described above allows to understand the difference between various logics in the class $Spec(I, N)$. Any model A of a logic $L \in Spec(I, N)$ is constructed from models of its counterparts, $\mathsf{A}^{\perp}$ and $\mathsf{A}_{\perp}$, and any element of A has two components, say, positive and negative, element of $\mathsf{A}^{\perp}$ and element of $\mathsf{A}_{\perp}$. As we can see from Proposition 22, the second component may impose restrictions on the admissible values of the first component. The nature of such restrictions characterizes the place of a logic in the interval $Spec(I, N)$. For all models of the upper point of the interval, $I \cap N$, any combination of positive and negative components is admissible. In models of the least logic, $I * N$, conversely, all possible restrictions, which the negative component may impose on the positive one, are admissible.

7. Cardinality of intervals *Spec*(I, N)

First, we improve the results of the previous section about the structure of intervals of the form $Spec(I, N)$.

For any logic $L \in \mathbf{Int}(\mathbf{Neg})$, we denote by $(L)\uparrow$ the class of its extensions in **Int** (respectively, in **Neg**), i.e., $(L)\uparrow = [L, \mathbf{Lk}]$ (respectively, $(L)\uparrow = [L, \mathbf{Lmn}]$).

Proposition 25. *For any $I \in \mathbf{Int}$ and $N \in \mathbf{Neg}$, the mapping $(L_1, L_2) \mapsto (L_1 * L_2) \cap (I \cap N)$ defines an isomorphic embedding of the direct product $(I)\uparrow \times (N)\uparrow$ into the interval $Spec(I, N)$.*

Proof. Let $Mod_{si}(L)$ denote the set of strongly compact models of a logic L. It is clear that $L_1 \cap L_2 = L(Mod_{si}(L_1) \cup Mod_{si}(L_2))$ for any two logics $L_1, L_2 \in \mathbf{Jhn}$. The variety of j-algebras is congruence distributive, therefore, by the well-known Jónsson's result $Mod_{si}(L_1 \cap L_2)$ is contained in the class $\mathsf{HSUp}(Mod_{si}(L_1) \cup Mod_{si}(L_2))$, where $\mathsf{H}(\mathsf{K})$ denotes the class of homomorphic images of algebras from the class K, $\mathsf{S}(\mathsf{K})$ denotes the class of subalgebras, and $\mathsf{Up}(\mathsf{K})$ the class of ultraproducts.

Let us consider an ultraproduct $\mathsf{B} := \prod_{i \in I} \mathsf{A}_i / F$, where each of the algebras A_i belongs $Mod_{si}(L_1) \cup Mod_{si}(L_2)$. Since F is an ultrafilter, one of the sets

$$\{i \mid \mathsf{A}_i \in Mod_{si}(L_1)\} \quad \text{or} \quad \{i \mid \mathsf{A}_i \in Mod_{si}(L_2)\}$$

belongs to F. Consequently, either $\mathsf{B} \models L_1$, or $\mathsf{B} \models L_2$. The class of models of a logic is obviously closed under homomorphic images and subalgebras, whence, any

$\mathsf{A} \in Mod_{si}(L_1 \cap L_2)$ is a model of L_1 or of L_2. We have thus proved that $Mod_{si}(L_1 \cap L_2) = Mod_{si}(L_1) \cup Mod_{si}(L_2)$.

Take logics $I_1, I_2 \in (I)\uparrow$ and $N_1, N_2 \in (N)\uparrow$ such that $(I_1, N_1) \neq (I_2, N_2)$. If $I_1 \neq I_2$ and, for definiteness, $I_1 \nsubseteq I_2$, let A be a strongly compact model of I_2 such that $\mathsf{A} \not\models I_1$ and let B be an arbitrary model of N_1. In case when $I_1 = I_2$, $N_1 \neq N_2$, and say $N_1 \nsubseteq N_2$, we let A to be an arbitrary strongly compact model of I_1, and B a model of N_2, but not of N_1.

Consider the j-algebra $\mathsf{B} \oplus \mathsf{A}$. The second greatest element of A will be also the second greatest element of $\mathsf{B} \oplus \mathsf{A}$, whence $\mathsf{B} \oplus \mathsf{A}$ is strongly compact. By Corollary 20, $\mathsf{B} \oplus \mathsf{A} \in Mod_{si}(I_2 * N_2) \setminus Mod_{si}(I_1 * N_1)$.

Further, $Mod_{si}(I \cap N) = Mod_{si}(I) \cup Mod_{si}(N)$ as it was proved above, i.e., any element of $Mod_{si}(I \cap N)$ is either Heyting or negative algebra. In this way, $\mathsf{B} \oplus \mathsf{A} \notin Mod_{si}(I \cap N)$. Due to relations $Mod_{si}((I_i * N_i) \cap (I \cap N)) = Mod_{si}(I_i * N_i) \cup Mod_{si}(I \cap N)$, $i = 1, 2$, we conclude that $\mathsf{B} \oplus \mathsf{A}$ is a model of $(I_2 * N_2) \cap (I \cap N)$ and not of $(I_i * N_i) \cap (I \cap N)$. We have thus proved that the mapping $(L_1, L_2) \mapsto (L_1 * L_2) \cap (I \cap N)$ is one-to-one. That this is a homomorphism follows immediately from the definition. Finally, the fact that the image $(L_1 * L_2) \cap (I \cap N)$ belongs to $Spec(I, N)$ for $(L_1, L_2) \in (I)\uparrow \times (N)\uparrow$ follows from Proposition 9. □

Corollary 26. *For any $I \in \mathbf{Int}$ and $N \in \mathbf{Neg}$,*

$$\left|Spec(I, \mathbf{Ln})\right| = \left|Spec(\mathbf{Li}, N)\right| = 2^{\omega}.$$

The proof follows immediately from the previous proposition and the well-known fact that $|(\mathbf{Ln})\uparrow| = |(\mathbf{Li})\uparrow| = 2^{\omega}$.

The last proposition does not yet give a complete impression of how rich intervals $Spec(I, N)$ are. Even when logics I and N have only finitely many extensions, the interval $Spec(I, N)$ is infinite.

To prove further results we will apply the technique of Jankov's formulas suggested by Jankov [2,3] and modified by Ono [9] and Wroński [14,15]. We recall some basic elements of Jankov's method adopting it for j-algebras. For further details the reader may consult the works cited above.

Let $\mathsf{A} = \langle A, \vee, \wedge, \to, \bot, 1\rangle$ be a not more than countable and strongly compact j-algebra. For each element $a \in \mathsf{A}$, $a \neq \bot$, we attach a unique propositional variable p_a. Further, for any $a \in \mathsf{A}$, we attach a unique atomic formula Z_a as follows

$$Z_a := \begin{cases} p_a, & \text{if } a \neq \bot, \\ \bot, & \text{if } a = \bot. \end{cases}$$

A *diagram* $\Delta(\mathsf{A})$ of A is the following set of formulas

$$\begin{aligned} \Delta(\mathsf{A}) := {} & \bigl\{Z_{a\vee b} \to (Z_a \vee Z_b), (Z_a \vee Z_b) \to Z_{a\vee b} \mid a, b \in A\bigr\} \\ & \cup \bigl\{Z_{a\wedge b} \to (Z_a \wedge Z_b), (Z_a \wedge Z_b) \to Z_{a\wedge b} \mid a, b \in A\bigr\} \\ & \cup \bigl\{Z_{a\to b} \to (Z_a \to Z_b), (Z_a \to Z_b) \to Z_{a\to b} \mid a, b \in A\bigr\}. \end{aligned}$$

Let A be a finite j-algebra. Then $\Delta(\mathsf{A})$ is a finite set of formulas and we can define the *Jankov formula* of A by

$$J(\mathsf{A}) := \Bigl(\bigwedge \Delta(\mathsf{A})\Bigr) \to Z_{\star_{\mathsf{A}}},$$

where $(\bigwedge \Delta(\mathsf{A}))$ is the conjunction of all formulas in $\Delta(\mathsf{A})$. It is easy to see that $J(\mathsf{A}) \notin L\mathsf{A}$. Moreover, the following statement holds.

Lemma 27. *Let A be a finite and strongly compact j-algebra. For each j-algebra B, the following two conditions are equivalent*:

(1) $J(\mathsf{A}) \notin L\mathsf{B}$,
(2) A *is embeddable into a quotient algebra of* B.

For the proof, see e.g. [14].

A sequence $\{L_i\}_{i<\omega}$ of logics from **Jhn** is said to be *strongly independent* if $L_i \nsubseteq \bigvee_{i \neq j} L_j$ for each $i < \omega$.

The following two facts are natural generalizations of Proposition 1.2 and Lemma 1.4 of [13] to the class of extensions of minimal logic.

Proposition 28. *Let $\{L_i\}_{i<\omega}$ be a strongly independent sequence of logics from* **Jhn**. *For every subsets I and J of ω, $I = J$ if and only if $\bigvee_{i\in I} L_i = \bigvee_{i \in J} L_i$.*

Proposition 29. *Suppose $\{\mathsf{A}_i\}_{i<\omega}$ is a sequence of strongly compact j-algebras satisfying the following conditions*:

(1) *each A_i is finite,*
(2) *for every $i, j < \omega$, $i \neq j$ implies that A_i cannot be embedded into any quotient algebra of A_j.*

If a logic L is contained in every $L\mathsf{A}_i$ $(i < \omega)$, the sequence $\{L_i\}_{i<\omega}$ of logics defined by $L_i = L + \{J(\mathsf{A}_i)\}$ $(i < \omega)$ is strongly independent.

Now we are ready to turn to the study of cardinalities of intervals $Spec(I, N)$.

Proposition 30. *The interval $Spec(\mathbf{Lk}, \mathbf{Lmn})$ has the following structure*:

$$\mathbf{Lk} * \mathbf{Lmn} \subseteq \cdots \subseteq L_n \subseteq \cdots \subseteq L_1 \subseteq L_0,$$

where $L_0 = \mathbf{Le} = \mathbf{Lk} \cap \mathbf{Lmn}$; $L_n = L(\mathsf{B}_n \oplus \mathbf{2}) = \mathbf{Lk} * \mathbf{Lmn} + \{J(\mathsf{B}_{n+1} \oplus \mathbf{2})\}$ for $n > 0$, where B_n is a negative Peirce algebra with n atoms (see Fig. 1); finally, $\mathbf{Lk} * \mathbf{Lmn} = L(\mathsf{B} \oplus \mathbf{2})$, where B is an arbitrary infinite negative Peirce algebra.

Proof. All logics from the interval $Spec(\mathbf{Lk}, \mathbf{Lmn})$ have the same negative models, namely, the models of **Lmn**. Therefore, every logic $L \in Spec(\mathbf{Lk}, \mathbf{Lmn})$ is determined by the class of its non-negative finitely generated strongly compact models, and we denote this class by $Mod^{+}_{fgsc}(L)$. Let $\mathsf{A} \in Mod^{+}_{fgsc}(L)$. Then $\mathsf{A}^{\perp}$ is non-trivial and strongly compact, moreover, $\mathsf{A}^{\perp} \models \mathbf{Lk}$. Any strongly compact Boolean algebra is two-element, therefore, A have the form $\mathsf{B} \oplus \mathbf{2}$, where B is a finitely generated model of **Lmn**, i.e., a finitely generated negative Peirce algebra. It is clear that $\mathsf{B} \oplus \mathbf{2}$ is finitely generated if and only if B is finitely generated.

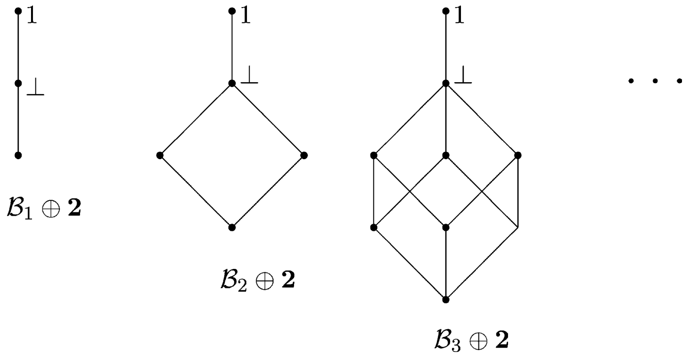

Fig. 1.

The only difference between Boolean and negative Peirce algebras is in the interpretation of $\perp$. In Boolean algebras, $\perp$ is interpreted as the least element, whereas in negative Peirce algebras, it is interpreted as the greatest element of an algebra. In this way, all finitely generated negative Peirce algebras are finite. And we have a countable chain of different (up to isomorphism) finitely generated negative Peirce algebras $\{\mathsf{B}_n \mid n \in \omega\}$, where B_n is a negative Peirce algebra with n-atoms, i.e., with n minimal elements in the set $\mathsf{B}_n \setminus \{f\}$ for f denoting the least element of B_n. Clearly, B_n is isomorphically embedded into B_m or is isomorphic to a homomorphic image of B_m if and only if $n \leqslant m$. Therefore, the class $Mod^+_{fgsc}(L)$ for $L \in Spec(\mathbf{Lk}, \mathbf{Lmn})$ may have only the form

$$M_\alpha = \{\mathsf{B}_n \oplus \mathbf{2} \mid n < \alpha\},$$

where $1 \leqslant \alpha \leqslant \omega$. It is not hard to check that each of these sets can be realized as a set of non-negative finitely generated strongly compact models of a suitable logic L_α from the interval $Spec(\mathbf{Lk}, \mathbf{Lmn})$. Indeed, the class $M_1 = \{\mathsf{B}_0 \oplus \mathbf{2}\} = \{\mathbf{2}\}$ corresponds to the logic $\mathbf{Le} = \mathbf{Lk} \cap \mathbf{Lmn}$, which has the unique non-negative strongly compact model, $\mathbf{2}$. For $1 < \alpha < \omega$, we consider a logic

$$L_\alpha := \mathbf{Lk} * \mathbf{Lmn} + \bigl\{J(\mathsf{B}_\alpha \oplus \mathbf{2})\bigr\}.$$

According to Lemma 27 the class $Mod^+_{fgsc}(L_\alpha)$ coincides with the class of algebras of the form $\mathsf{B}_n \oplus \mathbf{2}$ which are not embeddable into quotient algebras of $\mathsf{B}_\alpha \oplus \mathbf{2}$. Every proper quotient algebra of $\mathsf{B}_\alpha \oplus \mathbf{2}$ is negative and non-negative algebra cannot be embedded into it. The algebra $\mathsf{B}_n \oplus \mathbf{2}$ is embeddable into $\mathsf{B}_m \oplus \mathbf{2}$ if and only if B_n is embeddable into B_m. Combining all these facts we obtain

$$Mod^+_{fgsc}(L_\alpha) = M_\alpha.$$

From the last equality, we have $L_\alpha = L(\mathsf{B}_\alpha \oplus \mathbf{2})$.

The class $Spec(\mathbf{Lk}, \mathbf{Lmn})$ has the least element $L_\omega = \mathbf{Lk} * \mathbf{Lmn}$ and, obviously, $Mod^+_{fgsc}(\mathbf{Lk} * \mathbf{Lmn}) = M_\omega$. If we take an arbitrary infinite negative Peirce algebra B, any algebra of the form $\mathsf{B}_n \oplus \mathbf{2}$ will be embeddable into $\mathsf{B} \oplus \mathbf{2}$. Therefore, $L_\omega = L(\mathsf{B} \oplus \mathbf{2})$.

Classes of models M_α, $1 \leqslant \alpha \leqslant \omega$, form an ascending chain of type $\omega + 1$ with respect to inclusion, which means that the corresponding logics $\mathbf{L}_\alpha$, $1 \leqslant \alpha \leqslant \omega$, will form a descending chain of type $(\omega + 1)^*$. $\square$

One can find a rather weak condition (as compared to the analogous sufficient condition of Proposition 29) on finite models of negative counterpart N, which guaranties that for any intuitionistic counterpart the interval $Spec(I, N)$ will have the power of the continuum.

Proposition 31. *Let $I \in \mathbf{Int}$, $N \in \mathbf{Neg}$, and let there exist a family*

$$\{\mathsf{B}_i \mid i < \omega\}$$

of finite negative algebras such that $\mathsf{B}_i \models N$ for all $i < \omega$, and B_i is not embeddable into B_j for $i \neq j$. Then

$$\left|Spec(I, N)\right| = 2^{\omega}.$$

Proof. In view of Proposition 15 it is enough to consider the case $I = \mathbf{Lk}$. Consider the sequence $\{\mathsf{A}_i\}_{i<\omega}$ of j-algebras defined by $\mathsf{A}_i = \mathsf{B}_i \oplus \mathbf{2}$, $i < \omega$. It will be a sequence of strongly compact j-algebras modeling $\mathbf{Lk} * N$. By assumption if $i \neq j$, B_i is not embeddable into B_j, in which case also A_i is not embeddable into A_j. Further, note that every proper quotient algebra of A_j will be negative. This means that the non-negative algebra A_i cannot be embedded into a quotient algebra of A_j.

Define a sequence of logics $\{L_i\}_{i<\omega}$ by $L_i = \mathbf{Lk} * N + \{J(\mathsf{A}_i)\}$. By Proposition 29 this sequence is strongly independent, and we obtain a continuum of different logics of the form $\bigvee_{i\in I} L_i$, $I \subseteq \omega$. It can be easily proved that each of these logics belongs to the interval $Spec(\mathbf{Lk}, N)$. Indeed, on one hand, A_i is not a model for L_i and A_j, $j \neq i$, models L_i, therefore, L_i is nontrivial and nonnegative. This means that $(\mathbf{Lk} * N)_{int} \subseteq (L_i)_{int} = \mathbf{Lk}$. On the other hand, the second greatest element of A_i is $\perp$ and the formula $J(\mathsf{A}_i)$ has the form $\neg\varphi$, from which we infer

$$(L_i)_{neg} = \mathbf{Lk} * N + \{\perp, \neg\varphi\} = \mathbf{Lk} * N + \{\perp\} = (\mathbf{Lk} * N)_{neg} = N. \qquad \square$$

8. Kripke semantics

In conclusion, we will say a few words about analogues of upper and lower algebras in Kripke frames. The detailed explanation of Kripke semantics for extensions of minimal logic can be found in [12].

We call *Kripke j-frame*, or simply *j-frame*, a triple $\mathsf{W} = \langle W, \sqsubseteq, Q\rangle$, where W is a set of possible worlds, $\sqsubseteq$ is an accessibility relation such that $\langle W, \sqsubseteq\rangle$ is an ordinary Kripke frame for intuitionistic logic, i.e., a partially ordered set, and $Q \subseteq W$ is a cone (upward closed set) with respect to $\sqsubseteq$, which we will call the cone of *abnormal worlds*. Worlds lying out of Q are called *normal*. As usual, a *valuation* v of a j-frame W is a mapping from the set of propositional variables to the set of cones of the ordering $\langle W, \sqsubseteq\rangle$. A *model* $\mu = \langle \mathsf{W}, v\rangle$ is a pair consisting of a j-frame and its valuation. We say also in this case that μ is a model on W.

The forcing relation between models and formulas is defined in just the same way as for ordinary Kripke frames. The only exception is the case of constant $\perp$. More precisely, we define the relation $\mu \models_x \varphi$, where $\mu = \langle \mathsf{W}, v\rangle$ is a model, $\mathsf{W} = \langle W, \sqsubseteq, Q\rangle$, $x \in W$, and

φ is a formula, by induction on the structure of formulas as follows. For a propositional variable p_i, we put

$$\mu \models_x p_i \quad \Leftrightarrow \quad x \in v(p_i).$$

And further,

$$\mu \models_x \varphi \wedge \psi \quad \Leftrightarrow \quad \mu \models_x \varphi \text{ and } \mu \models_x \psi;$$
$$\mu \models_x \varphi \vee \psi \quad \Leftrightarrow \quad \mu \models_x \varphi \text{ or } \mu \models_x \psi;$$
$$\mu \models_x \varphi \to \psi \quad \Leftrightarrow \quad \forall y \in W \quad \big(x \sqsubseteq y \;\Rightarrow\; (\mu \models_y \varphi \;\Rightarrow\; \mu \models_y \psi)\big).$$

Finally, for the constant $\bot$, we put

$$\mu \models_x \bot \quad \Leftrightarrow \quad x \in Q.$$

In particular, for a negated formula $\neg\varphi$ considered as an abbreviation for $\varphi \to \bot$, we have

$$\mu \models_x \neg\varphi \quad \Leftrightarrow \quad \forall y \in W \quad \big(x \sqsubseteq y \;\Rightarrow\; (\mu \models_y \varphi \;\Rightarrow\; y \in Q)\big).$$

We will read $\mu \models_x \varphi$ as "a formula φ is true at a world (or at a point) x in a model μ". As usual, we say that a formula φ is *true on a model* $\mu = \langle \mathsf{W}, v\rangle$, $\mu \models \varphi$, if for all $x \in W$ the relation $\mu \models_x \varphi$ holds. A formula φ is *true on a j-frame* W, $\mathsf{W} \models \varphi$, if it is true on a model $\langle \mathsf{W}, v\rangle$ for an arbitrary valuation v of the j-frame W. A formula φ is *valid on the class* K of Kripke j-frames if $\mathsf{W} \models \varphi$ for any j-frame $\mathsf{W} \in \mathsf{K}$.

Let $\mathsf{W} = \langle W, \sqsubseteq, Q\rangle$ be a j-frame and let $K \subseteq W$ be a cone with respect to $\sqsubseteq$. We define a j-frame W^K in the following way $\mathsf{W}^K := \langle K, \sqsubseteq^K, Q^K\rangle$, where $\sqsubseteq^K = \sqsubseteq \cap (K)^2$, $Q^K = Q \cap K$. If $\mu = \langle \mathsf{W}, v\rangle$ is a model on W, then $\mu^K = \langle \mathsf{W}^K, v^K\rangle$, where $v^K(p) = v(p) \cap K$ for all propositional variables p.

Lemma 32. *Let* $\mathsf{W} = \langle W, \sqsubseteq, Q\rangle$ *be an arbitrary j-frame, μ a model on* W*, and* $K \subseteq W$ *a cone. For any $x \in K$ and an arbitrary formula φ, we have*

$$\mu \models_x \varphi \quad \Leftrightarrow \quad \mu^K \models_x \varphi.$$

In particular,

$$\mathsf{W} \models \varphi \quad \Rightarrow \quad \mathsf{W}^K \models \varphi.$$

We say that a j-frame W is a *model for* a logic $L \in \mathbf{Jhn}$, $\mathsf{W} \models L$, if $\mathsf{W} \models \varphi$ for all $\varphi \in L$. For a class K of frames, we put

$$L\mathsf{K} := \{\varphi \mid \mathsf{W} \models \varphi \text{ for any } \mathsf{W} \in \mathsf{K}\}.$$

If $L = L\mathsf{K}$ for some $L \in \mathbf{Jhn}$, we will say that L is characterized by the class K.

We will call a j-frame $\mathsf{W} = \langle W, \sqsubseteq, Q\rangle$ *normal* if $Q = \emptyset$, i.e., if all worlds of this frame are normal. It is clear that normal j-frames can be identified with ordinary Kripke frames for intuitionistic logic. A j-frame $\mathsf{W} = \langle W, \sqsubseteq, Q\rangle$ is *abnormal* if $Q = W$, i.e., if all worlds are abnormal. Finally, a j-frame $\mathsf{W} = \langle W, \sqsubseteq, Q\rangle$ will be called *identical* if the accessibility relation $\sqsubseteq$ coincides with the identity relation on W, $\sqsubseteq = Id_W$.

The end-point logics of the intervals **Int**, **Neg**, and **Par** can be characterized by the following classes of j-frames.

Proposition 33 [12].

(1) *Minimal logic* **Lj** *is characterized by the class of all j-frames.*
(2) *Intuitionistic logic* **Li** *is characterized by the class of all normal j-frames.*
(3) *Minimal negative logic* **Ln** *is characterized by the class of all abnormal j-frames.*
(4) *Logic of classical refutability* **Le** *is characterized by the class of all identical j-frames.*
(5) *Classical logic* **Lk** *is characterized by the class of all identical normal j-frames.*
(6) *Maximal negative logic* **Lmn** *is characterized by the class of all identical abnormal j-frames.*

For an arbitrary j-frame $\mathsf{W} = \langle W, \sqsubseteq, Q\rangle$ we define the following frames

$$\mathsf{W}^{(+)} := \big\langle W \setminus Q, \sqsubseteq \cap (W \setminus Q)^2, \emptyset\big\rangle \quad \text{and} \quad \mathsf{W}^{(-)} := \mathsf{W}^Q = \langle Q, \sqsubseteq \cap Q^2, Q\rangle.$$

Obviously, the frame $\mathsf{W}^{(+)}$ is normal, whereas $\mathsf{W}^{(-)}$ is abnormal for an arbitrary j-frame W. As we can see from the propositions below, the frames defined above are analogues of upper and lower algebras associated with the given j-algebra.

First of all we mention the following simple fact. For any j-frame W and any formula φ, the translation $In(\varphi)$ is true on the j-frame $\mathsf{W}^{(-)}$,

$$\mathsf{W}^{(-)} \models In(\varphi).$$

This fact can be checked via an easy induction on the structure of formulas.

Lemma 34. *Let W be an arbitrary j-frame, v a valuation of $\mathsf{W}^{(+)}$, and let v' be any valuation of W such that for any propositional variable p we have $v(p) = v'(p) \cap (W \setminus Q)$. Then for any formula φ and for an arbitrary element $x \in W \setminus Q$ the following equivalence holds*

$$\langle \mathsf{W}, v'\rangle \models_x In(\varphi) \quad \Leftrightarrow \quad \langle \mathsf{W}^{(+)}, v\rangle \models_x \varphi.$$

Proof. Let $\mu' := \langle \mathsf{W}, v'\rangle$ and $\mu^{(+)} := \langle \mathsf{W}^{(+)}, v\rangle$.

We argue by induction on the structure of formulas. The case of $\perp$ is trivial. For an arbitrary propositional variable p and $x \in W \setminus Q$ we have $\mu' \models_x p \vee \perp$ if and only if $x \in v'(p)$. Thus, we have $x \in v'(p)$ and $x \in W \setminus Q$, i.e., $x \in v(p)$. The latter is equivalent to $\mu^{(+)} \models_x p$.

Now, we assume that for formulas φ and ψ, and for all $x \in W \setminus Q$ the equivalences

$$\mu' \models_x In(\varphi) \Leftrightarrow \mu^{(+)} \models_x \varphi \quad \text{and} \quad \mu' \models_x In(\psi) \Leftrightarrow \mu^{(+)} \models_x \psi$$

hold. Prove that the desired equivalence takes place for the implication $\varphi \to \psi$.

Let $\mu' \models_x In(\varphi \to \psi)$ $(= In(\varphi) \to In(\psi))$ for some $x \in W \setminus Q$. This means that for all $x \sqsubseteq y \in W$ the relation $\mu' \models_y In(\varphi)$ implies $\mu' \models_y In(\psi)$. In view of the assumed equivalences, we have

$$\forall y \in W \setminus Q\big(x \sqsubseteq y \;\Rightarrow\; (\mu^{(+)} \models_y \varphi \;\Rightarrow\; \mu^{(+)} \models_y \psi)\big),$$

and so $\mu^{(+)} \models_x \varphi \to \psi$.

Conversely, let $\mu^{(+)} \models_x \varphi \to \psi$ for some $x \in W \setminus Q$. Taking into account our assumption, we have that for all $x \sqsubseteq y \in W \setminus Q$ if $\mu' \models_y In(\varphi)$, then $\mu' \models_y In(\psi)$. If $y \in Q$, then $\mu' \models_y In(\varphi)$ and $\mu' \models_y In(\psi)$. Thus, for all $x \sqsubseteq y \in W$,

$$\mu' \models_y In(\varphi) \;\Rightarrow\; \mu' \models_y In(\psi),$$

which means that $\mu' \models_x In(\varphi \to \psi)$.

The cases of disjunction and conjunction are trivial. □

Proposition 35. *For a j-frame W and a formula φ, the following equivalences hold*

$$\mathsf{W} \models In(\varphi) \quad\Leftrightarrow\quad \mathsf{W}^{(+)} \models \varphi,$$
$$\mathsf{W} \models \bot \to \varphi \quad\Leftrightarrow\quad \mathsf{W}^{(-)} \models \varphi.$$

Proof. The first equivalence immediately follows from the previous lemma and the remark before lemma.

If $\mathsf{W} \models \bot \to \varphi$, then for any valuation v of W, the formula φ is true in all abnormal worlds of the model $\langle \mathsf{W}, v\rangle$, which means by Lemma 32 that $\langle \mathsf{W}^{(-)}, v^Q\rangle \models \varphi$. Any valuation v of $\mathsf{W}^{(-)}$ can be considered as a valuation of W, in which case $v = v^Q$. Thus, for all valuations v of $\mathsf{W}^{(-)}$, we have $\langle \mathsf{W}^{(-)}, v\rangle \models \varphi$, i.e., $\mathsf{W}^{(-)} \models \varphi$.

Conversely, the assumption $\mathsf{W}^{(-)} \models \varphi$ implies that for any valuation v of W, $\langle \mathsf{W}^{(-)}, v^Q\rangle \models \varphi$. In view of Lemma 32, the latter means that for all valuations v of W, the formula φ will be true at any abnormal world of a model $\langle \mathsf{W}, v\rangle$, which implies, in turn, $\langle \mathsf{W}, v\rangle \models \bot \to \varphi$. □

The following statement easily follows from Propositions 10 and 35.

Corollary 36. *Let $L \in$ **Jhn** and $\mathsf{W} \models L$. Then*

$$\mathsf{W}^{(+)} \models L_{int} \quad \textit{and} \quad \mathsf{W}^{(-)} \models L_{neg}.$$

For a class of j-frames K we define

$$\mathsf{K}^{(+)} := \{\mathsf{W}^{(+)} \mid \mathsf{W} \in \mathsf{K}\}, \qquad \mathsf{K}^{(-)} := \{\mathsf{W}^{(-)} \mid \mathsf{W} \in \mathsf{K}\}.$$

Proposition 37. *Let K be a class of j-frames and let $L = L\mathsf{K}$. Then $L_{int} = L\mathsf{K}^{(+)}$ and $L_{neg} = L\mathsf{K}^{(-)}$.*

Proof. The inclusion $L_{int} \subseteq L\mathsf{K}^{(+)}$ follows from Corollary 36. We argue for the inverse inclusion. Take a $\varphi \notin L_{int}$, in which case $In(\varphi) \notin L$. Consequently, there exist a frame $\mathsf{W} \in \mathsf{K}$, its valuation v, and an element $x \in W$ such that $\langle \mathsf{W}, v\rangle \not\models_x In(\varphi)$. As was noticed above, any formula of the form $In(\psi)$ is true in any model at any abnormal element, therefore, $x \notin Q$. Whence, by Lemma 34, we have $\mathsf{W}^{(+)} \not\models \varphi$.

Now we turn to the second equality. Again, we have to prove only the inclusion $L_{neg} \subseteq L\mathsf{K}^{(-)}$ since the inverse inclusion follows from Corollary 36. Let φ does not belong to L_{neg}, i.e., $L \nvdash \bot \to \varphi$. Consider a j-frame $\mathsf{W} \in \mathsf{K}$ such that $\mathsf{W} \not\models \bot \to \varphi$. From the last relation we obtain by Proposition 35 that $\mathsf{W}^{(-)} \not\models \varphi$, i.e., $\varphi \notin L\mathsf{K}^{(-)}$. □

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